

# More on scattering of Chern-Simons vortices

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## Abstract

I derive a general formalism for finding kinetic terms of the effective Lagrangian for slowly moving Chern-Simons vortices. Deformations of fields linear in velocities are taken into account. From the equations they must satisfy I extract the kinetic term in the limit of coincident vortices. For vortices passing one over the other there is locally the right-angle scattering. The method is based on analysis of field equations instead of action functional so it may be useful also for nonvariational equations in nonrelativistic models of Condensed Matter Physics.

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## 1 Introduction

There is quite a large family of field-theoretical models both relativistic [1, 2, 3, 5, 4, 6, 12, 11] and nonrelativistic [7, 8, 10, 9] with solitons which possess Bogomol'nyi [13] limit. In this limit static field equations in a given topological sector can be reduced for a minimal energy configuration to first order differential equations. These equations generically admit static multisoliton solutions characterised by a finite set of parameters such as positions of solitons and their internal orientations. The configurations are static - if we think about solitons as about particles there are no net static forces between them.

It was an idea of Manton [14] that low energy scattering of monopoles in the Bogomol'nyi-Prasad-Somerfield model [13, 6] can be modeled by reduction of the dynamics to a finite-dimensional manifold of parameters of static multisoliton solutions. The kinetic part of the Lagrangian of the original theory after integrating out spacial dependence of the fields with by now time-dependent parameters yields the kinetic part of effective Lagrangian quadratic in time derivatives of parameters. A metric on the moduli space can be read out of it. A fundamental idea in this approach is that configurations satisfying Bogomol'nyi lower bound are at a bottom of a potential well. A slow motion of solitons can lead only to small deformations of fields with respect to static configurations.

The idea was successfully applied to scattering of monopoles, vortices in Abelian Higgs model [4, 15, 17, 16],  $CP^1$  solitons [18, 5]. Recently also extensions of the method to the case of Chern-Simons vortices both relativistic and nonrelativistic were done [19, 20, 21, 22, 23, 24]. However as was first pointed out in [20] in these cases a new problem arises. Lagrangians of these models contain terms linear in time derivatives such as Chern-Simons term and/or Schrodinger action. By just promoting parameters to the role of collective coordinates one can reliably calculate only terms in the effective Lagrangian linear in velocities. To compute kinetic part one has to take into account small deformations of the fields with respect to static configurations. It is enough to consider only deformations linear in velocities. Such deformations can be in principle calculated from full field equations linearised in deformations and terms linear in velocities. However also terms linear in accelerations and third time derivatives arise and as I have discussed in [22] there is no apparent reason why they could be negligible as compared to velocities. Such an "approximation" can lead to serious inconsistencies.

In this paper I put the problem on a slightly different footing. The acceleration terms are not neglected. There are no net static forces between vortices so their accelerations must be zero for vanishing velocities. Thus we can assume that acceleration vector is

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at least linear in velocities. We can expect this linear term to be nonzero because there are charge-flux interactions by Lorenz-like forces. Thus acceleration is not neglected but replaced by a position-dependent matrix  $\omega$  times velocity. The same procedure can be applied iteratively to the third time derivative. Finally in a special limit of coinciding vortices such a unique form of  $\omega$ -matrix is extracted which admits regular deformations of fields. A knowledge of  $\omega$  is enough to establish the form of moduli space metrics. A brief comment on how this approach works in Abelian Higgs model is added.

## 2 The model, zero modes and useful notations

We take the Lagrangian of the self-dual Chern-Simons-Higgs model in the form

$$L = \frac{\kappa}{2} \varepsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha + D_\mu \phi^* D^\mu \phi - V(|\phi|) , \quad (1)$$

where  $V(|\phi|) = \frac{1}{\kappa^2} (|\phi|^2 - 1)^2$ ,  $D_\mu \phi = \partial_\mu \phi - A_\mu \phi$ , the signature of the flat 2+1 dimensional metrics is  $(+, -, -)$  and the Levi-Civita symbol is chosen so that  $\varepsilon^{012} = -1$ . A variation of the action with respect to  $A_0$  leads to Gauss' law constraint

$$2\phi^* \phi (\partial_t \chi - A_0) = \kappa B , \quad (2)$$

where we have introduced  $\phi = |\phi| e^{i\chi}$  and the magnetic field is  $B = -F_{12}$ . When one takes into account the Gauss' law one can effectively rewrite the original Lagrangian in a new form

$$L = f^2 + \kappa B \partial_t \chi + \frac{\kappa}{2} \varepsilon^{ij} A_i \partial_t A_j - [\partial_i f \partial_i f + f^2 (\partial_i \chi - A_i)^2 + \frac{\kappa^2 B^2}{4f^2} + \frac{1}{\kappa^2} f^2 (f^2 - 1)^2] . \quad (3)$$

We have just introduced  $f = |\phi|$  and  $\varepsilon^{ij}$  such that  $\varepsilon^{12} = 1$ . The energy density for a static configuration with a positive topological index is

$$\varepsilon = [\partial_i f - f \varepsilon_{ik} (\partial_k \chi - A_k)]^2 + \frac{1}{f^2} \left[ \frac{\kappa B}{2} + \frac{f^2 (1 - f^2)}{\kappa} \right]^2 - B . \quad (4)$$

The magnetic flux is quantised as  $2\pi n$  with  $n$  being the winding number. Thus energy is bounded from below by this value of magnetic flux. If  $\kappa > 0$  this Bogomol'nyi lower bound is saturated by static configurations being solutions to first order differential equations

$$\partial_i f = f \varepsilon_{ik} (\partial_k \chi - A_k) , \quad (5)$$

$$\varepsilon_{ij} \partial_i A_j = \frac{2}{\kappa^2} f^2 (1 - f^2) , \quad (6)$$

$$A_0 = \frac{1 - f^2}{\kappa} . \quad (7)$$

These equations admit static multivortex solutions parametrised by a set of  $2n$  real parameters. In Coulomb gauge one can take  $\chi = \sum_{v=1}^n \text{Arg}(z - z_v)$ , where the sum runs over vortices labeled by  $v$ 's and complex coordinates on the plane are used. In this gauge there is only one second order differential equation to solve

$$\nabla^2 \ln f = \frac{2}{\kappa^2} f^2 (f^2 - 1) + 2\pi \sum_{v=1}^n \delta^{(2)}(z - z_v) . \quad (8)$$

Once  $f$  is known other fields can be obtained from Eqs.(2,5). The singular sources on the R.H.S. of the above equation enforce modulus  $f$  in a close vicinity of  $p$ -fold zero  $z_0$  to behave like  $|z - z_0|^p$  (it is a leading term of the expansion). This equation in particular admits cylindrically symmetric vortex solution with winding number  $n$   $\phi = f(r) \exp in\theta$ . We will parametrise multivortex solutions in following equivalent forms

$$\phi = (z - z_1) \dots (z - z_n) W(z, \bar{z}, z_v) \equiv (z^n - \sum_{k=0}^{n-1} \lambda_k z^k) W(z, \bar{z}, z_v) \quad (9)$$

and

$$\chi = \frac{1}{2i} \ln \frac{\prod_{v=1}^n (z - z_v)}{\prod_{w=1}^n (\bar{z} - \bar{z}_w)} \equiv \frac{1}{2i} \ln \frac{z^n - \sum_{k=0}^{n-1} \lambda_k z^k}{\bar{z}^n - \sum_{l=0}^{n-1} \bar{\lambda}_l \bar{z}^l} . \quad (10)$$

$\lambda$ 's are complex coefficients of the  $n$ -th degree polynomial with roots  $z_v$  and  $W$  is a positive real function. Although the parametrisations are equivalent it will later appear that  $\lambda$ 's are more efficient in description of vortices passing one over another.

Each of these multivortex solutions possesses  $2n$  zero modes.

$$\begin{aligned}
\delta f(z, \lambda) &= \frac{\partial f}{\partial \lambda_p^A}(z, \lambda) \delta \lambda_p^A \equiv f(z, \lambda) h_p^A(z, \lambda) \delta \lambda_p^A, \\
\delta \chi(z, \lambda) &= \frac{\partial \chi}{\partial \lambda_p^A} \delta \lambda_p^A = -Im\left(\frac{l^A z^p}{z^n - \lambda_s z^s}\right) \delta \lambda_p^A, \\
\delta A_k &= \partial_k \delta \chi + \varepsilon_{kl} \partial_l \frac{\delta f}{f} = (\partial_k \frac{\partial \chi}{\partial \lambda_p^A} + \varepsilon_{kl} \partial_l h_p^A) \delta \lambda_p^A, \\
\delta A_0 &= -\frac{2}{\kappa^2} f^2 h_p^A \delta \lambda_p^A,
\end{aligned} \tag{11}$$

where  $\lambda_p = \lambda_p^1 + i\lambda_p^2$  and  $l^A = (1, i)$ . Equation (8) linearised in fluctuations becomes

$$\nabla^2 h_p^A + \frac{4}{\kappa^2} \rho(1 - 2\rho) h_p^A = 0. \tag{12}$$

$\rho$  denotes moduli squared  $f^2$ . From the asymptotics of  $f$  close to its zeros we can extract the leading term in fluctuations

$$h_p^A \sim -Re\left(\frac{l^A z^p}{z^n - \lambda_s z^s}\right) \tag{13}$$

as  $(z^n - \lambda_s z^s) \sim 0$ . This is the only singular term in the expansion around the actual zero of the Higgs field. This singularity is fine-tuned by the singularity in  $\delta \chi$  to yield regular  $\delta A_k$  (see (11)).

Now because of future applications let us take a closer look at the coincident  $n$ -vortex solution  $\phi = f(r) \exp in\theta$  with  $f(r)$  satisfying

$$f f'' + \frac{f f'}{r} - (f')^2 = \frac{2}{\kappa^2} f^4 (f^2 - 1) \tag{14}$$

with boundary conditions  $f(0) = 0$  and  $f(\infty) = 1$ . Close to zero it behaves like  $f \sim f_0 r^n - \frac{f_0^3}{2\kappa^2(n+1)^2} r^{3n+2} + \dots$ . Fluctuations for small  $\lambda$ 's can be written as

$$h(r, \theta) = h_p^A(r, \theta) \lambda_p^A = -H_{n-p} [\lambda_{n-p}^1 \cos(n-p)\theta + \lambda_{n-p}^2 \sin(n-p)\theta]. \tag{15}$$

$H$ 's satisfy following equations

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(n-p)^2}{r^2}\right) H_{n-p} + \frac{4}{\kappa^2} \rho(1 - 2\rho) H_{n-p} = 0, \tag{16}$$

with a normalisation that close to zero  $H_m \sim \frac{1}{r^m}$  and it asymptotically vanishes at infinity.

### 3 Slowly moving vortices

For more clarity in this paragraph and in what follows we will rescale gauge fields  $A_\mu \rightarrow \kappa^{-1} A_\mu$  and coordinates  $x^\mu \rightarrow \kappa^{-1} x^\mu$ . On the level of field equations it amounts to fixing  $\kappa = +1$ .

The aim of this paper is to investigate slow motion of vortices in adiabatic approximation. In the case of Chern-Simons solitons a new difficulty arises because there are terms linear in time derivatives in the Lagrangian of the theory. Only terms linear in velocities can be correctly calculated by direct application of former methods. As was pointed out in [20] to obtain the kinetic term one has not only to make the static fields time-dependent by time-variation of their  $2n$  parameters but one also has to take into account deformations of the "static" fields up to terms linear in velocities. It is something like a generalised Lorenz transformation. In this paragraph we will develop general formalism for calculating such corrections. It will appear that to have regular solutions one can not neglect terms proportional to accelerations as it was anticipated in [22].

Let us make the fields time dependent by variation of parameters and add to them deformations linear in velocities, say  $f(z, \lambda) \rightarrow f[z, \lambda(t)] + \Delta f[z, \lambda(t), \dot{\lambda}(t)]$  etc. Evaluation of the effective Lagrangian goes by substitution of such fields into Lagrangian (3) and integrating out their planar dependence. The only terms which can contribute to the part of the effective Lagrangian linear in velocities are

$$L_{eff}^{(1)} = \int d^2x [B \dot{\chi} + \frac{1}{2} \varepsilon^{ij} A_i \dot{A}_j]. \tag{17}$$

and one needs to take into account only effects of promoting parameters to the role of collective coordinates. All the other terms are quadratic in velocities. The contribution of the second term in Eq.(17) vanishes in Coulomb gauge. For fairly separated vortices one can approximate under the integral  $B = -2\pi \sum_v \delta^{(2)}(z - z_v)$  to obtain

$$L_{eff}^{(1)} \approx -2\pi \frac{d}{dt} \sum_{v>w} Arg(z_v - z_w). \tag{18}$$

With the explicite form of the phase (10) and some integration by parts one can rewrite the linear Lagrangian as

$$L_{eff}^{(1)} = -2\pi \sum_v \dot{x}_v^i A_i(z_v) , \quad (19)$$

where  $z_v = x_v^1 + ix_v^2$ .

The second order term of the effective Lagrangian is

$$\begin{aligned} L_{eff}^{(2)} &= \dot{f}^2 + 2\dot{f}\dot{\Delta}f + \dot{\Delta}f\dot{\Delta}f - (\partial_i \Delta f)^2 - \frac{1}{2}V''(f)(\Delta f)^2 \\ &+ f^2(\dot{\chi} - \Delta A_0)^2 - f^2(\Delta A_i)^2 + (\Delta f)^2 A_0^2 - (\Delta f)^2(\partial_i \chi - A_i)^2 \\ &+ \Delta A_0 \Delta B + \frac{1}{2}\varepsilon_{ij}(2\dot{A}_i \Delta A_j + \dot{\Delta}A_i \Delta A_j) \\ &- 4f\Delta f A_0(\dot{\chi} - \Delta A_0) + 4f\Delta f(\partial_i \chi - A_i)\Delta A_i . \end{aligned} \quad (20)$$

The small corrections to the fields has to be calculated from following equations obtained by linearisation of full field equations.

$$\begin{aligned} \nabla^2\left(\frac{\Delta f}{f}\right) + 4\rho(2-3\rho)\left(\frac{\Delta f}{f}\right) + (\partial_k \ln \rho)[\partial_k\left(\frac{\Delta f}{f}\right) + \varepsilon_{kl}(\Delta A_l - \partial_l \Delta \chi)] - 2(\rho-1)(\Delta A_0 - \dot{\chi}) &= \frac{\ddot{f}}{f} + 2(\rho-1)\dot{\Delta}\chi + \frac{\ddot{\Delta}f}{f} \\ \nabla^2 \Delta \chi - \partial_k \Delta A_k - 2(2\rho-1)\frac{\dot{f}}{f} + (\partial_k \ln \rho)(\varepsilon_{kl}\partial_l \frac{\Delta f}{f} + \partial_k \Delta \chi - \Delta A_k) &= \ddot{\chi} - \dot{\Delta}A_0 + \ddot{\Delta}\chi - 2(1-\rho)\frac{\dot{\Delta}f}{f} \\ \varepsilon_{kl}\partial_k \Delta A_l - 4f(1-\rho)\Delta f + 2\rho(\dot{\chi} - \Delta A_0) &= -2\rho\dot{\Delta}\chi \\ \partial_n \Delta A_0 + 2\partial_n \rho \frac{\Delta f}{f} + 2\rho\varepsilon_{nk}(\partial_k \Delta \chi - \Delta A_k) - \partial_n \dot{\chi} - \varepsilon_{nk}\partial_k\left(\frac{\dot{f}}{f}\right) &= \dot{\Delta}A_n . \end{aligned} \quad (21)$$

The equations were simplified with a use of static field equations satisfied by background fields. Now the crucial observation is that in the Bogomol'nyi limit any forces exerted on vortices must be zero for vanishing velocities so accelerations are at least linear in velocities

$$\ddot{\lambda}_p^A = \omega_{pq}^{AB}(\lambda)\dot{\lambda}_q^B . \quad (22)$$

$\omega$  for given  $p$  is a matrix in indices  $A, B$ . This relation can be iterated to give

$$\frac{d^3}{dt^3}\lambda_p^A = \omega_{pq}^{AB}(\lambda)\ddot{\lambda}_q^B + \frac{\partial\omega_{pq}^{AB}}{\partial\lambda_r^C}\dot{\lambda}_q^B\dot{\lambda}_r^C \approx \omega_{pq}^{AB}\omega_{qr}^{BC}\dot{\lambda}_r^C , \quad (23)$$

where once again we have preserved only terms linear in velocities. We can also make following definitions and approximations

$$\begin{aligned} \dot{f} &= fh_p^A\dot{\lambda}_p^A , \\ \ddot{f} &\approx fh_p^A\omega_{pq}^{AB}\dot{\lambda}_q^B , \\ \Delta f &\equiv fs_p^A\dot{\lambda}_p^A , \\ \dot{\Delta}f &\approx fs_p^A\omega_{pq}^{AB}\dot{\lambda}_q^B , \\ \ddot{\Delta}f &\approx fs_p^A\omega_{pr}^{AC}\omega_{rq}^{CB}\dot{\lambda}_q^B , \\ \dot{\chi} &\approx \frac{\partial\chi}{\partial\lambda_p^A}\omega_{pq}^{AB}\dot{\lambda}_q^B , \\ \Delta A_k &\equiv A_k^{Ap}\dot{\lambda}_p^A , \\ \Delta A_0 &\equiv a_p^A\dot{\lambda}_p^A , \end{aligned} \quad (24)$$

With these formulas and with the gauge  $\Delta\chi = 0$  Eqs.(21) can be rewritten as

$$\begin{aligned} \dot{\lambda}_p^A[\nabla^2 s_p^A + 4\rho(2-3\rho)s_p^A + (\partial_k \ln \rho)(\partial_k s_p^A + \varepsilon_{kl}A_l^{Ap}) - 2(\rho-1)(a_p^A - \frac{\partial\chi}{\partial\lambda_p^A})] &= h_p^A\omega_{pq}^{AB}\dot{\lambda}_q^B + s_p^A\omega_{pr}^{AC}\omega_{rq}^{CB}\dot{\lambda}_q^B \\ \dot{\lambda}_p^A[-\partial_k A_k^{Ap} - 2(2\rho-1)h_p^A + (\partial_k \ln \rho)(\varepsilon_{kl}\partial_l s_p^A - A_k^{Ap})] &= [\frac{\partial\chi}{\partial\lambda_p^A} - a_p^A - 2(1-\rho)s_p^A]\omega_{pq}^{AB}\dot{\lambda}_q^B \\ \dot{\lambda}_p^A[\varepsilon_{kl}\partial_k A_l^{Ap} - 4\rho(1-\rho)s_p^A + 2\rho(\frac{\partial\chi}{\partial\lambda_p^A} - a_p^A)] &= 0 \\ \dot{\lambda}_p^A[\partial_n a_p^A + 2\partial_n \rho s_p^A - 2\rho\varepsilon_{nk}A_k^{Ap}] - (\partial_n \frac{\partial\chi}{\partial\lambda_p^A} + \varepsilon_{nk}\partial_k h_p^A) &= A_n^{Ap}\omega_{pq}^{AB}\dot{\lambda}_q^B . \end{aligned} \quad (25)$$

They should yield field deformations in approximation up to terms linear in velocities. In the limit of very small velocities field deformations become small as compared to background fields so our linearisations of field equations with respect to deformations are justified. Because full field deformations are regular functions then in the limit of slow motion also Eqs.(25) must have regular solutions being good approximations to full deformations. With this in mind we can take for granted existence of regular solutions and derive necessary conditions for the regularity. The unknown parameters in Eqs.(25) are elements of matrices  $\omega_{pq}^{AB}(\lambda)$ . Our strategy from now on is to adjust such unique values of these parameters which allow solutions to be regular. Once we know the parameters we will also know equations of motion for  $\lambda$ 's linearised in velocities

$$\ddot{\lambda}_p^A = \omega_{pq}^{AB} \dot{\lambda}_q^B . \quad (26)$$

Since we know linear part of the effective Lagrangian (18) the knowledge of these equations enables us to restore also its quadratic part up to total derivatives. The general form of the Lagrangian is

$$L_{eff} = g_{pq}^{AB}(\lambda) \dot{\lambda}_p^A \dot{\lambda}_q^B + b_{pq}^{AB}(\lambda) \lambda_p^A \dot{\lambda}_q^B . \quad (27)$$

The metric tensor on the moduli space  $g$  must be symmetric under exchange of pairs of indices  $(A, p)$  and  $(B, q)$  and invertible

$$(g^{-1})_{pq}^{AB} g_{qr}^{BC} = \delta^{AC} \delta_{pr} . \quad (28)$$

Equations of motion linearised in velocities are

$$2g_{pq}^{AB} \ddot{\lambda}_p^A + (b_{pq}^{AB} - b_{qp}^{BA}) \dot{\lambda}_p^A + \left( \frac{\partial b_{pq}^{AB}}{\partial \lambda_s^D} - \frac{\partial b_{ps}^{AD}}{\partial \lambda_q^B} \right) \lambda_p^A \dot{\lambda}_s^D . \quad (29)$$

The components of the metric tensor must solve following equations

$$g_{qr}^{BC} \omega_{rp}^{CA} + \frac{1}{2}(b_{pq}^{AB} - b_{qp}^{BA}) + \frac{1}{2} \left( \frac{\partial b_{rq}^{CB}}{\partial \lambda_p^A} - \frac{\partial b_{rp}^{CA}}{\partial \lambda_q^B} \right) \lambda_r^C = 0. \quad (30)$$

This is a system of  $2n \times 2n$  linear inhomogenous equations. The basic condition for the system to have an unique solution is that matrix  $\omega$  is invertible in pairs of indices  $(A, p)$  and  $(C, r)$ ,  $\det \omega \neq 0$ . This means (see Eq.(26)) that whatever is the small velocity it is always a source of acceleration already in linear terms. It should be a quite generic case except some anomalous sets of measure zero in a model with magnetic interaction. The metrics can be extended to these exceptional points by continuity. One of such points is certainly the limit of infinitely separated vortices where magnetic interaction degenerates to a purely topological term (Eq.(18)). But in this limit the effective Lagrangian can be accurately calculated with a help of the product Ansatz of independently Lorenz-boosted vortices. The quadratic term reads

$$L_{eff}^{(2)} \approx \pi \sum_v \dot{z}_v \dot{z}_v^* . \quad (31)$$

The second and third terms in Eq.(30) are explicitly antisymmetric under exchange of pairs  $(A, p)$  and  $(B, q)$  so the first term also has to be antisymmetric  $\omega^T g = -g\omega$ . This condition means that to leading order acceleration is orthogonal to velocity with respect to metric  $g$ . Once again it should be so for forces due to magnetic interactions and they are the only forces linear in velocities. If this condition is satisfied we are left with  $2n^2 - n$  independent equations necessary to establish the same number of metric tensor's components.

## 4 Two vortices in center of mass frame

We will consider the by now classic example of two vortices in CM frame. The system is well described by two parameters  $\lambda_1$  and  $\lambda_2$  which can be identified with former notations

$$\lambda_1 = \lambda_0^1 , \quad \lambda_2 = \lambda_0^2 . \quad (32)$$

The Higgs field is  $\phi = (z^2 - \lambda)W(z, \bar{z}, \lambda)$ . Eqs.(25) are on both sides linear in velocities so they can be cast in a form of a velocity-independent matrix multiplying a vector of velocities. For the product to be always zero any velocity vector must belong to kernels of the matrices. A matrix is zero if and only if it annihilates any vector from the basis spanning the space of velocities. If we choose  $\dot{\lambda}_1 \neq 0$  and  $\dot{\lambda}_2 = 0$  Eqs.(25) reduce to

$$\begin{aligned} \nabla^2 s^{(1)} + 4\rho(2 - 3\rho)s^{(1)} + (\partial_k \ln \rho)(\partial_k s^{(1)} + \varepsilon_{kl} A_l^{(1)}) - 2(\rho - 1)(a^{(1)} - \frac{\partial \chi}{\partial \lambda^{(1)}}) &= h^{(A)} \omega^{A1} + s^{(B)} \omega^{BA} \omega^{A1} \\ -\partial_k A_k^{(1)} - 2(2\rho - 1)h^{(1)} + (\partial_k \ln \rho)(\varepsilon_{kl} \partial_l s^{(1)} - A_k^{(1)}) &= [\frac{\partial \chi}{\partial \lambda^{(A)}} - a^{(A)} - 2(1 - \rho)s^{(A)}] \omega^{A1} \\ \varepsilon_{kl} \partial_k A_l^{(1)} - 4\rho(1 - \rho)s^{(1)} + 2\rho(\frac{\partial \chi}{\partial \lambda_1} - a^{(1)}) &= 0 \\ \partial_n a^{(1)} + 2\partial_n \rho s^{(1)} - 2\rho \varepsilon_{nk} A_k^{(1)} - (\partial_n \frac{\partial \chi}{\partial \lambda^{(1)}} + \varepsilon_{nk} \partial_k h^{(1)}) &= A_n^{(A)} \omega^{A1} . \end{aligned} \quad (33)$$

On the other hand for the choice  $\dot{\lambda}_1 = 0$  and  $\dot{\lambda}_2 \neq 0$  we obtain

$$\begin{aligned}
\nabla^2 s^{(2)} + 4\rho(2 - 3\rho)s^{(2)} + (\partial_k \ln \rho)(\partial_k s^{(2)} + \varepsilon_{kl} A_l^{(2)}) - 2(\rho - 1)(a^{(2)} - \frac{\partial \chi}{\partial \lambda^{(2)}}) &= h^{(A)} \omega^{A2} + s^{(B)} \omega^{BA} \omega^{A2} \\
-\partial_k A_k^{(2)} - 2(2\rho - 1)h^{(2)} + (\partial_k \ln \rho)(\varepsilon_{kl} \partial_l s^{(2)} - A_k^{(2)}) &= [\frac{\partial \chi}{\partial \lambda^{(A)}} - a^{(A)} - 2(1 - \rho)s^{(A)}] \omega^{A2} \\
\varepsilon_{kl} \partial_k A_l^{(2)} - 4\rho(1 - \rho)s^{(2)} + 2\rho(\frac{\partial \chi}{\partial \lambda_2} - a^{(2)}) &= 0 \\
\partial_n a^{(2)} + 2\partial_n \rho s^{(2)} - 2\rho \varepsilon_{nk} A_k^{(2)} - (\partial_n \frac{\partial \chi}{\partial \lambda^{(2)}} + \varepsilon_{nk} \partial_k h^{(2)}) &= A_n^{(A)} \omega^{A2} .
\end{aligned} \tag{34}$$

These two sets of equations have to be satisfied simultaneously.

Now let us consider the limit of coincident vortices. Such a configuration is rotationally symmetric and it should be all the same what is the direction in which it is splitting. This motivates the limiting form of the effective Lagrangian

$$L_{eff}^{\lambda \rightarrow 0} \sim h_0 \frac{1}{2} |\lambda|^\xi \delta^{AB} \dot{\lambda}_A \dot{\lambda}_B - g_0 \varepsilon^{AB} \lambda_A \dot{\lambda}_B , \tag{35}$$

with the coefficient  $h_0$  and the power  $\xi$  to be determined.  $g_0$  can be extracted with a help of Eq.(17) and fluctuations (11,15)

$$g_0 = 4\pi \int_0^\infty dr \frac{\rho(1 - 2\rho)H_2(r)}{r} \tag{36}$$

A value of  $g_0$  was estimated numerically to be  $g_0 \approx 0.0194$ .

Motivated by null powers of  $\lambda$ 's in Eqs.(32,33) and by

$$\frac{\partial \chi}{\partial \lambda_1} = \frac{\sin 2\theta}{r^2} , \quad \frac{\partial \chi}{\partial \lambda_2} = -\frac{\cos 2\theta}{r^2} \tag{37}$$

in the limit of vanishing  $\lambda$ 's we can restrict field deformations to following forms in polar coordinates  $r, \theta$

$$\begin{aligned}
s^{(1)} &= s(r) \sin 2\theta , \quad s^{(2)} = -s(r) \cos 2\theta , \\
a^{(1)} &= a(r) \sin 2\theta , \quad a^{(2)} = -a(r) \cos 2\theta , \\
A_r^{(1)} &= b(r) \cos 2\theta , \quad A_r^{(2)} = b(r) \sin 2\theta , \\
A_\theta^{(1)} &= c(r) \sin 2\theta , \quad A_\theta^{(2)} = -c(r) \cos 2\theta .
\end{aligned} \tag{38}$$

together with a form of the matrix  $\omega^{AB} = \omega \varepsilon^{AB}$ . Eqs.(33) and (34) reduce to

$$\begin{aligned}
s'' + \frac{s'}{r} - \frac{4s}{r^2} + \omega^2 s + 4\rho(2 - 3\rho)s + \frac{\rho'}{\rho}(s' + c) - 2(\rho - 1)(a - \frac{1}{r^2}) &= \omega H_2 , \\
b' + \frac{b}{r} + \frac{2c}{r} - 2(2\rho - 1)H_2 + \frac{\rho'}{\rho}(b - \frac{2s}{r}) &= -\frac{\omega}{r^2} + \omega a - 2\omega(\rho - 1)s , \\
c' + \frac{c}{r} + \frac{2b}{r} - 4\rho(1 - \rho)s + 2\rho(\frac{1}{r^2} - a) &= 0 , \\
a' + 2\rho's - 2\rho c + \omega b &= -\frac{2}{r^3} + \frac{2H_2}{r} , \\
\frac{2a}{r} + 2\rho b - \omega c &= \frac{2}{r^3} + H_2' .
\end{aligned} \tag{39}$$

$\omega$  is an adjustable parameter we have to choose in such a way that solutions are regular. Short inspection shows that fourth equation in the above set of equations can be derived from the fifth, second and third. Regularity of  $\Delta f$  means that  $s$  can not be more divergent then  $O(r^{-2})$ . Thus we can expand the regular solution around  $r = 0$  as

$$s(r) = \sum_{k=-2}^{\infty} s_k r^k , \quad a(r) = \sum_{k=0}^{\infty} a_k r^k , \quad b(r) = \sum_{k=1}^{\infty} b_k r^k , \quad c(r) = \sum_{k=1}^{\infty} c_k r^k . \tag{40}$$

Substitution of first few terms in the expansions to Eqs.(39) shows that they are in contradiction unless  $\omega = -2$ . If we adopt this value of  $\omega$  following leading terms will be obtained

$$\begin{aligned}
s &= (-v - \xi)r^2 + \dots , \\
a &= (-v - 2\xi)r^2 + \dots , \\
b &= (-2v - 2\xi)r + \dots , \\
c &= (2v + 2\xi)r ,
\end{aligned} \tag{41}$$

where  $v \approx 1387000$  is a coefficient in the expansion  $H_2 = r^{-2} + vr^2 + \dots$  and  $\xi$  is a free parameter coming from a "homogenous" part of the solution. For the solution regular at infinity numerical analysis has given  $\xi \approx -1387017.5$ .

With the value of  $\omega = -2$  we can conclude that the limiting form of the effective Lagrangian for  $\lambda \rightarrow 0$  is

$$L_{eff}^{\lambda \rightarrow 0} = g_0 \left( \frac{1}{2} \delta^{AB} \dot{\lambda}_A \dot{\lambda}_B - \varepsilon^{AB} \lambda_A \dot{\lambda}_B \right) . \quad (42)$$

In a generic case of two vortices in CM frame at  $(R, \Theta)$  and  $(R, \Theta + \pi)$  the effective Lagrangian must take a form

$$L_{eff} = F(R) \dot{R}^2 + 2G(R) \dot{R} \dot{\Theta} + H(R) R^2 \dot{\Theta}^2 + B(R) R \dot{\Theta} . \quad (43)$$

The functions  $F, G, H$  are to be determined. It is a general form of the metrics tensor invariant with respect to rotations. The function  $G$  is in general nonzero since the Chern-Simons term breaks parity invariance. Eq.(20) also contains terms which break parity and there does not seem to be any reason why these terms should vanish. In polar coordinates the  $\omega$ -matrix reads

$$\omega^{AB} = \frac{\frac{d}{dR}[RB(R)]}{2R} (g^{-1})^{AC} \varepsilon^{CB} = \frac{J'(R)}{2R} (g^{-1})^{AC} \varepsilon^{CB} \quad (44)$$

with  $J(R)$  being the total spin of two vortices separated by a distance  $2R$ . It is an invertible matrix and acceleration is indeed to leading order orthogonal to velocity. I have attempted calculating  $\omega$  but because there is less symmetry (less constraints) in the problem for generic  $R$  then for  $R = 0$  it can not be extracted just from the asymptotics close to zeros of the Higgs field. Matching with asymptotics at infinity would be necessary.

Finally a comment on analogous scattering of vortices in Abelian Higgs model (AHM) is in order. In this model  $G(R) = 0$  and also  $B(R) = 0$ . Similar considerations as just above lead to a conclusion that  $\omega^{AB} = 0$ . Thus in AHM accelerations are at least quadratic in velocities. In equations analogous to (21) terms linear in acceleration and its time derivative can be neglected as compared to those linear in velocities. These terms were indeed neglected in Appendix B of Ref.[20] and it was shown that rearrangement of effective Lagrangian analogous to (20) with a help of equations fulfilled by deformation leads to the same expression as that derived by Samols [15]. Here we have shown justification of these steps when performed in AHM.

Let us consider a general Bogomol'nyi theory and try to decide what are the conditions for the effective Lagrangian to be purely quadratic in time derivatives. A Lagrangian of a theory can be written as

$$L = G_{ab}[\psi] \dot{\psi}_a \dot{\psi}_b + K_a[\psi] \dot{\psi}_a - \varepsilon[\psi] \quad (45)$$

where  $\psi$ 's are a set of fields,  $\varepsilon[\psi]$  is a static energy density functional and  $G_{ab}[\psi]$  is an invertible, symmetric and positively definite tensor. The only contribution to the linear part of the effective Lagrangian is

$$L_{eff}^{(1)} = K_a[\psi] \dot{\psi}_a , \quad (46)$$

where like in all of this paper  $\psi$ 's are the fields of the static self-dual background and the time derivative means a total derivative with respect to time-dependent parameters. This term certainly vanishes if  $K_a[\psi] = 0$  for the given background. More subtle possibility is that the whole expression (46) can yield zero result when its spatial dependence is integrated out. Relativistic gauge theory contains linear terms like in Eq.(45). In the Abelian Higgs model such a term is like  $-\partial_i A_0 \dot{A}_i + e\psi^* \psi A_0 \dot{\chi}$  but the self-dual background has the property that  $A_0 = 0$  and that is why there is no linear term in the effective Lagrangian. An interesting example of Maxwell-Higgs self-dual model with relativistic kinetics can be found in [11]. A uniform background charge density forces nonzero  $A_0$ . Vortices in this model feel both Magnus force and mutual magnetic interactions.

The quadratic term of the general effective Lagrangian is

$$L_{eff}^{(2)} = G_{ab}[\psi] (\dot{\psi}_a \dot{\psi}_b + 2\dot{\psi}_a \Delta \dot{\psi}_b + \Delta \dot{\psi}_a \Delta \dot{\psi}_b) + \frac{\delta K_a}{\delta \psi_b} (\dot{\psi}_a \Delta \psi_b - \Delta \psi_a \dot{\psi}_b - \Delta \psi_a \Delta \dot{\psi}_b) - \frac{\delta^2 \varepsilon}{\delta \psi_a \delta \psi_b} [\psi] \Delta \psi_a \Delta \psi_b , \quad (47)$$

where we have introduced time derivatives of background fields and deformations of fields. Now we restrict to the case of  $L_{eff}^{(1)} = 0$ . This means that accelerations are at least quadratic in velocities and the above formula can be reduced to

$$L_{eff}^{(2)} = G_{ab}[\psi] \dot{\psi}_a \dot{\psi}_b + \frac{\delta K_a}{\delta \psi_b} (\dot{\psi}_a \Delta \psi_b - \Delta \psi_a \dot{\psi}_b) - \frac{\delta^2 \varepsilon}{\delta \psi_a \delta \psi_b} [\psi] \Delta \psi_a \Delta \psi_b , \quad (48)$$

We can linearise field equations with respect to velocities. The field deformations must satisfy a simple equation

$$-\frac{\delta^2 \varepsilon}{\delta \psi_a \delta \psi_b} \Delta \psi_b = \left( \frac{\delta K_a}{\delta \psi_b} - \frac{\delta K_b}{\delta \psi_a} \right) \dot{\psi}_b . \quad (49)$$

This relation enables us to simplify Eq.(48) to the following compact form

$$L_{eff} = G_{ab}[\psi] \dot{\psi}_a \dot{\psi}_b \quad (50)$$

plus higher order terms negligible in adiabatic approximation.

Thus whenever there is no linear term in the effective Lagrangian the quadratic term can be correctly calculated just with a help of background fields with their parameters promoted to the role of collective coordinates.

## 5 Conclusions

The limiting form of the term quadratic in velocities of the effective Lagrangian for two Chern-Simons vortices was extracted from equations satisfied by deformations of the fields with respect to static background. This form shows that as we trace locally trajectories of vortices passing one over another there is the celebrated right-angle scattering. Globally if two vortices were pushed from a large distance one against the other with zero impact parameter they would avoid direct collision their trajectories being curved by charge-flux interactions. A difference between a total spin of the pair of vortices when they are infinitely separated and when they sit on top of one another is  $-2\pi$  (for  $\kappa = 1$ ). Thus the necessary condition for the zeros to meet and the right-angle scattering to occur is that an impact parameter  $d$  with respect to the center of mass and an initial velocity  $v$  satisfy  $dv = \frac{1}{2}$ . This condition can become sufficient only for  $d$  small enough because vortex-vortex magnetic interactions are falling exponentially with a distance.

The local right angle scattering is a hint that moduli-space manifold is similar to a smoothed cone. The missing volume can show itself in a thermodynamics of a vortex gas by an excluded volume in van der Waals state equation similarly as for the vortex gas in Abelian Higgs model [25].

Finally let me stress that problems partially overcome in this paper are not at all special to relativistic Chern-Simons models. They can also appear in nonrelativistic and nonvariational models so celebrated in Condensed Matter Physics because Schrodinger or diffusion terms are linear in time derivatives on one hand and on the other hand the effective action method may be not able to describe the whole variety of dynamical phenomena. I would like to address such problems in a near future.

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